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## Noether's theorem and time-dependent quantum invariants\*

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**Abstract.** The time-dependent integrals of motion, linear in position and momentum operators, of a quantum system are extracted from Noether's theorem prescription by means of special time-dependent variations of coordinates. For the stationary case of the generalized two-dimensional harmonic oscillator, the time-independent integrals of motion are shown to correspond to special Bragg-type symmetry properties. A detailed study for the non-stationary case of this quantum system is presented. The linear integrals of motion are constructed explicitly for the case of varying mass and coupling strength. They are also obtained from Noether's theorem. The general treatment for a multi-dimensional quadratic system is indicated and it is shown that the time-dependent variations that give rise to the linear invariants, as conserved quantities, satisfy the corresponding classical homogeneous equations of motion for the coordinates.

### 1. Introduction

For some stationary and non-stationary systems the time-dependent integrals of motion have been constructed explicitly [1–5]. On the other hand it is well known that the integrals of the motion of classical and quantum systems are related to the symmetry of the system. This relation is expressed by Noether's theorem [6, 7]. The generalized stationary two-dimensional oscillator has been analysed from the point of view of Noether's theorem [8]. In this work based on Noether's theorem the time-independent integrals of the motion that are polynomials in the position and momentum operators were found.

Lewis and Riesenfeld [1] constructed integrals of motion quadratic in position and momentum operators for a time-dependent one-dimensional oscillator and for a charged spinless particle moving in a homogeneous varying magnetic field. In [2] the time-dependent integrals of motion that are linear in position and momentum operators have been found for a charged particle moving in a time-dependent homogeneous magnetic field. In [3] the linear time-dependent integrals of motion have been obtained for the quantum forced oscillator with time-dependent frequency. Using these integrals of motion the dynamical symmetries  $SU(2)$  and  $SU(1, 1)$  [3] have been associated with the charged particle moving in a varying magnetic field and the parametric quantum oscillator. In [4] the time-dependent invariants, linear in position and momentum operators, have been found for multi-dimensional quantum systems the Hamiltonian of which is a non-stationary general quadratic form in position and momentum operators. These integrals of motion have been used to construct the propagator, coherent states and transition amplitudes between the energy levels of the system in terms of symplectic transformations  $ISp(2N, \mathbb{R})$  [4, 5]. The time-dependent integrals of motion give

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a useful method for finding the propagator of quantum systems using the system of equations found in [9] (see also [5]). Urrutia and Hernández [10] considered a non-stationary damped harmonic oscillator, and have proved that the relation of linear time-dependent integrals of motion to the propagator is close in spirit to the Schwinger action principle method. In [11], and more recently [12], the connection between time-dependent integrals of motion of some examples of stationary physical problems and the dynamical symmetry concept of quantum systems has been emphasized.

On the other hand, up to now it was not clear how the time-dependent integrals of motion found in [1–5] could be obtained from the canonical procedure of Noether's approach nor what symmetry corresponds to linear time-dependent integrals of motion of non-stationary multi-dimensional forced harmonic oscillator. The aim of the present work is to show that such a variation in coordinates for the generalized oscillator [8] as well as for the non-stationary one- and multi-dimensional oscillators for which Lagrangian variation is reduced to a total time-derivative term exist. So from normal Noether's theorem we will obtain linear time-dependent integrals of the motion for the generalized two-dimensional oscillator and connect these integrals with the analogue of the Bragg scattering relation giving the time-independent invariants found in [8]. We will also consider explicitly the non-stationary generalized two-dimensional oscillator with varying mass and a time-dependent coupling term proportional to the third component of the angular momentum operator. It is also known that time-dependent Hamiltonians [5] can generate squeezed states and, therefore, we are going to study the two-mode squeezing [13–15] in the framework of this model. Finally we show that the analysis made for the generalized two-dimensional oscillator can be extended for multi-dimensional non-stationary quadratic systems. Thus we find its corresponding time-dependent invariants, linear in position and momentum, by means of Noether's theorem.

## 2. Generalized two-dimensional harmonic oscillator

Following [8] we will recall in this section the properties of the integrals of motion and Noether's theorem for the two-dimensional generalized oscillator. We will start from the time-independent system

$$H = \frac{1}{2} \sum_{i=1}^2 (p_i^2 + x_i^2) + \lambda M \quad (2.1)$$

where we are using dimensionless units, i.e.  $\hbar = m_0 = \omega_0 = 1$ . This Hamiltonian is constructed from a two-dimensional harmonic oscillator plus  $M = x_1 p_2 - x_2 p_1$ , the projection of the angular momentum in the  $O_z$  direction, with coupling constant  $\lambda$ . For  $\lambda = 1$ , the Hamiltonian (2.1) describes the Landau problem of a charged particle moving in a constant magnetic field. Introducing the variables

$$a_i^\dagger = \frac{1}{\sqrt{2}}(x_i - ip_i) \quad a_i = \frac{1}{\sqrt{2}}(x_i + ip_i) \quad i = 1, 2 \quad (2.2)$$

and

$$\eta_\pm = \frac{1}{\sqrt{2}}(a_1^\dagger \pm ia_2^\dagger) \quad \xi_\pm = \frac{1}{\sqrt{2}}(a_1 \mp ia_2) \quad (2.3)$$

which obey commutation relations

$$[\xi_a, \eta_b] = \delta_{ab} \quad [\xi_a, \xi_b] = [\eta_a, \eta_b] = 0 \quad a, b = +, - \quad (2.4)$$

we can rewrite the Hamiltonian (1.1) in terms of these operators

$$H = (1 + \lambda)N_+ + (1 - \lambda)N_- \tag{2.5}$$

where a constant term has been neglected and  $N_a$ ,  $a = \pm$ , denotes the number of quanta in direction  $a$ . If  $|\lambda| > 1$ , Hamiltonian (2.1) has an energy spectrum unbounded from below. This spectrum, except for constant terms, is described by the formula

$$E_{vm} = (1 + \lambda)n_+ + (1 - \lambda)n_- = v + \lambda m \tag{2.6}$$

with  $|m| = v, v - 2, \dots, 1$  or  $0$  and  $v = n_+ + n_-$ ,  $m = n_+ - n_-$ , and  $n_{\pm} = 0, 1, 2, \dots$ . The accidental degeneracy of the Hamiltonian (1.5) was explained in [8] by means of Noether's theorem and also the existence of time-independent integrals of motion. These constants of the motion, depending on the strength of the parameter  $\lambda$ , are given by

$$\eta_+^{k_1} \eta_-^{k_2} \quad \xi_+^{k_1} \xi_-^{k_2} \quad \text{for } |\lambda| > 1 \tag{2.7a,b}$$

$$\eta_+^{k_1} \xi_-^{k_2} \quad \xi_+^{k_1} \eta_-^{k_2} \quad \text{for } |\lambda| < 1 \tag{2.7c,d}$$

$$\eta_{\pm} \xi_{\pm} \quad \eta_{\mp} \xi_{\mp} \quad \text{for } \lambda = \mp 1 \tag{2.7e,f}$$

where the relative prime integers  $k_1$  and  $k_2$  are connected with the rational number  $\lambda$  by the formula

$$\frac{1 - \lambda}{1 + \lambda} = -\epsilon \frac{k_1}{k_2} \tag{2.8}$$

with the number  $\epsilon = -1$  for  $|\lambda| < 1$  and  $\epsilon = 1$  for  $|\lambda| > 1$ . The question which we want to answer here is how to find the time-dependent integrals of motion, linear in position and momentum, for the system and how these invariants are related to the integrals of motion (2.7). The system with Hamiltonian (2.5) is quadratic and due to the results in [2-5] there is a four-dimensional symplectic matrix defined by  $\Lambda$ , which relates the linear time-dependent integrals of motion with the position and momentum operators:

$$\begin{pmatrix} \pi_{10}(t) \\ \pi_{20}(t) \\ q_{10}(t) \\ q_{20}(t) \end{pmatrix} = \Lambda(t) \begin{pmatrix} \pi_1 \\ \pi_2 \\ q_1 \\ q_2 \end{pmatrix} \tag{2.9}$$

where we associate indices 1 and 2 with the labels  $+$  and  $-$ . Besides this matrix  $\Lambda(t)$  satisfies the first-order differential equation

$$\frac{d\Lambda}{dt} = \Lambda \Sigma B \tag{2.10}$$

with the initial condition  $\Lambda(0) = I_4$ .  $I_4$  denotes the  $4 \times 4$  identity matrix and the matrix  $\Sigma$  has the form

$$\Sigma = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \tag{2.11}$$

where  $I_2$  denotes the  $2 \times 2$  identity matrix.  $B$  is a matrix determined by the Hamiltonian (2.5), and, in the case considered, it is

$$B = \begin{pmatrix} 1 + \lambda & 0 & 0 & 0 \\ 0 & 1 - \lambda & 0 & 0 \\ 0 & 0 & 1 + \lambda & 0 \\ 0 & 0 & 0 & 1 - \lambda \end{pmatrix}. \tag{2.12}$$

Equation (2.10) may be easily integrated and the integrals of the motion which are linear forms in position and momentum may be written down in the form of creation and annihilation operators

$$A_1(t) = \exp\{i(1 + \lambda)t\}A_1(0) \quad A_2(t) = \exp\{i(1 - \lambda)t\}A_2(0) \quad (2.13)$$

where we choose the initial conditions

$$A_1(0) = \xi_+ \quad A_2(0) = \xi_- \quad (2.14)$$

Then, the invariants (2.7) are immediately obtained from the integrals of motion (2.13) by calculating the operators

$$A_1^{k_1}(t)A_2^{k_2}(t) \quad \text{for } |\lambda| > 1 \quad (2.15a)$$

$$A_1^{k_1}(t)A_2^{\dagger k_2}(t) \quad \text{for } |\lambda| < 1 \quad (2.15b)$$

$$A_1(t) = \xi_+ \quad \text{for } \lambda = -1 \quad (2.15c)$$

$$A_2(t) = \xi_- \quad \text{for } \lambda = 1. \quad (2.15d)$$

By substituting (2.13) and (2.14) into the relations (2.15a,b) and demanding that these integrals of motion do not depend on time, we obtain the conditions

$$k_1(1 + \lambda) + k_2(1 - \lambda) = 0 \quad \text{for } |\lambda| > 1 \quad (2.16a)$$

$$k_1(1 + \lambda) - k_2(1 - \lambda) = 0 \quad \text{for } |\lambda| < 1. \quad (2.16b)$$

These conditions recall the Bragg relation in x-ray crystallography. Thus for some integers  $k_1$  and  $k_2$  the dependence on time of the integrals of motion (2.15) disappears, in agreement with the accidental degeneracy implied by equation (2.8).

### 3. Linear invariants and Noether's theorem

We have found, in the previous section, the linear invariants (2.13) and (2.14) simply by guessing their form and proving that they are integrals of motion by direct checking. Now we will discuss how they can follow from Noether's theorem. Before doing this, to illustrate the procedure we consider first how the linear invariants for a one-dimensional parametric oscillator found in [2, 3] follow from Noether's theorem. The Lagrangian of this oscillator has the form

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2(t)q^2. \quad (3.1)$$

Following the Noether's theorem procedure used in [8] let us consider the variation in coordinate

$$\delta q = h(t). \quad (3.2)$$

We have used a specific variation depending only on an arbitrary time-dependent function  $h(t)$ , and it is straightforward to calculate the induced variation in the Lagrangian

$$\delta L = \dot{q}(\delta \dot{q}) - (\omega^2 q)\delta q = \dot{q}h'(t) - \omega^2 q h(t). \quad (3.3)$$

This variation can be written as a total time derivative of a function,  $\Omega = \dot{h}q$ , with respect to time if the function  $h(t)$  satisfies the equation

$$\ddot{h} + \omega^2(t)h = 0 \tag{3.4}$$

which is identical to the equation of motion of the physical system. The integrals of motion, due to the Noether theorem procedure, are determined by the function  $\Omega$  in the form

$$K(t) = \frac{\partial L}{\partial \dot{q}} \delta q - \Omega = h\dot{q} - \dot{h}q. \tag{3.5}$$

Considering trapped particle problems, Glauber [16] pointed out that the invariant of [2, 3] may be obtained by comparing it with the structure of a Wronskian. Since  $\dot{q} = p$ , expression (3.5) gives linear time-dependent integrals of motion. Therefore we have obtained these invariants from Noether's theorem.

Let us now return to the two-dimensional generalized oscillator (2.1) as another and more complicated example. The Lagrangian for this system has the form

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}m(\omega_0^2 - \lambda^2)(x_1^2 + x_2^2) + \lambda m(x_2\dot{x}_1 - x_1\dot{x}_2) \tag{3.6}$$

where, for later convenience, we take into account explicitly the mass and the frequency of the oscillator; however, if we put  $m = \omega_0 = 1$  we recover the case discussed in section 1. On the basis of experience with the one-dimensional problem let us consider a variation of coordinates  $x_1$  and  $x_2$  of the form

$$\delta x_1 = h_1(t) \quad \delta x_2 = h_2(t). \tag{3.7}$$

The corresponding induced variation in the Lagrangian (3.6) is expressed as

$$\delta L = m(\dot{h}_1 + \lambda h_2)\dot{x}_1 + m(\dot{h}_2 - \lambda h_1)\dot{x}_2 - m[\lambda \dot{h}_2 + (\omega_0^2 - \lambda^2)h_1]x_1 + m[\lambda \dot{h}_1 - (\omega_0^2 - \lambda^2)h_2]x_2.$$

It may be rewritten in the following manner:

$$\begin{aligned} \delta L = & \frac{d}{dt} \{m(\dot{h}_1 + \lambda h_2)x_1 + m(\dot{h}_2 - \lambda h_1)x_2\} \\ & - m \left[ \ddot{h}_1 + \frac{\dot{m}}{m}\dot{h}_1 + 2\lambda \dot{h}_2 + (\omega_0^2 - \lambda^2)h_1 + \frac{(\lambda m)'}{m}h_2 \right] x_1 \\ & - m \left[ \ddot{h}_2 + \frac{\dot{m}}{m}\dot{h}_2 - 2\lambda \dot{h}_1 + (\omega_0^2 - \lambda^2)h_2 - \frac{(\lambda m)'}{m}h_1 \right] x_2. \end{aligned} \tag{3.8}$$

The equations of motion for the generalized oscillator with Lagrangian (3.6) are

$$\begin{aligned} \ddot{x}_1 = & (\lambda^2 - \omega_0^2)x_1 - 2\lambda \dot{x}_2 - \frac{\dot{m}}{m}\dot{x}_1 - \frac{(\lambda m)'}{m}x_2 \\ \ddot{x}_2 = & (\lambda^2 - \omega_0^2)x_2 + 2\lambda \dot{x}_1 - \frac{\dot{m}}{m}\dot{x}_2 + \frac{(\lambda m)'}{m}x_1. \end{aligned} \tag{3.9}$$

Comparing the last two terms of (2.8) with (2.9), we conclude again that if the variations of coordinates  $h_1(t)$  and  $h_2(t)$  satisfy the system of equations of motion of the physical system the variation of the Lagrangian (2.6) takes the form of a full derivative  $\delta L = d\Omega/dt$ , with the function  $\Omega$  defined by

$$\Omega = m(\dot{h}_1 + \lambda h_2)x_1 + m(\dot{h}_2 - \lambda h_1)x_2. \tag{3.10}$$

Due to the Noether's theorem prescription we have the integrals of motion of the system

$$I(t) = \frac{\partial L}{\partial \dot{x}_1} h_1 + \frac{\partial L}{\partial \dot{x}_2} h_2 - \Omega.$$

These invariants are linear in positions and momenta

$$I(t) = (p_1 + \lambda m x_2) h_1 - m \dot{h}_1 x_1 + (p_2 - \lambda m x_1) h_2 - m \dot{h}_2 x_2. \tag{3.11}$$

Thus we explained from Noether's theorem the existence of linear time-dependent integrals of motion. There are four invariants because there exist four independent solutions for the system of equations (3.9). These different solutions are denoted by a superindex:  $h_1^{(k)}, h_2^{(k)}$ , with  $1 \leq k \leq 4$ . This set of constants of motion can be rewritten in the form (2.9), with the matrix  $\Lambda$  defined by the row vectors

$$\Lambda^{(k)} = (h_1^{(k)}, h_2^{(k)}, -m(\dot{h}_1^{(k)} + \lambda h_2^{(k)}), m(\lambda \dot{h}_1^{(k)} - \dot{h}_2^{(k)})). \tag{3.12}$$

The constants of motion, which we denote by  $p_{i0}$  and  $x_{i0}$ , satisfy the initial conditions  $p_{10}(0) = p_1, p_{20}(0) = p_2, x_{10}(0) = x_1$  and  $x_{20}(0) = x_2$ . These imply that the matrix  $\Lambda(0) = I_4$ , i.e.

$$\mathbf{h}_1(0) = (1, 0, 0, 0)^T \quad \dot{\mathbf{h}}_1(0) = (0, -\lambda(0), -\frac{1}{m(0)}, 0)^T \tag{3.13a}$$

$$\mathbf{h}_2(0) = (0, 1, 0, 0)^T \quad \dot{\mathbf{h}}_2(0) = (\lambda(0), 0, 0, -\frac{1}{m(0)})^T \tag{3.13b}$$

where  $\mathbf{h}_\alpha(t) = (h_\alpha^{(1)}, h_\alpha^{(2)}, h_\alpha^{(3)}, h_\alpha^{(4)})^T, \alpha = 1, 2$ . To find the explicit form of the variations  $h_1^{(k)}(t)$  and  $h_2^{(k)}(t)$ , we need to solve the classical equations of motion of the physical system. Let us introduce the change of variables

$$z = h_1 + i h_2 \quad z^* = h_1 - i h_2 \tag{3.14}$$

which allows us to rewrite the system of differential equations for  $h_1$  and  $h_2$  as

$$\ddot{z} + \left(\frac{\dot{m}}{m} - 2i\lambda\right) \dot{z} + \left[\omega_0^2 - \lambda^2 - \frac{i}{m}(\lambda m)\right] z = 0. \tag{3.15}$$

By means of the transformation

$$z(t) = m^{-1/2} \exp\left\{i \int_0^t d\tau \lambda(\tau)\right\} w(t) \tag{3.16}$$

equation (3.15) can be simplified to the form

$$\ddot{w} + \Omega^2 w = 0 \tag{3.17}$$

with

$$\Omega^2 = \omega_0^2 + \left(\frac{\dot{m}}{2m}\right)^2 - \frac{\ddot{m}}{2m}. \tag{3.18}$$

For appropriate choices of the time-dependent function  $\Omega$ , equation (3.17) can be solved and then from expressions (3.14) and (3.16) we have the general solution for (3.9):

$$\begin{aligned} h_1 &= \frac{1}{2\sqrt{m}} \left( w \exp\left\{i \int_0^t d\tau \lambda(\tau)\right\} + w^* \exp\left\{-i \int_0^t d\tau \lambda(\tau)\right\} \right) \\ h_2 &= \frac{1}{2i\sqrt{m}} \left( w \exp\left\{i \int_0^t d\tau \lambda(\tau)\right\} - w^* \exp\left\{-i \int_0^t d\tau \lambda(\tau)\right\} \right). \end{aligned} \tag{3.19}$$

To find the linear time-dependent integrals of motion of the generalized harmonic oscillator with constant parameters, we take  $m = \omega_0 = 1$  and  $\lambda$  an arbitrary constant. The solution of equation (3.17) is

$$w(t) = A \exp(it) + B \exp(-it).$$

From (3.19) we have the solutions for  $h_1$  and  $h_2$ :

$$h_1(t) = |A| \cos(t + \lambda t + \phi_A) + |B| \cos(t - \lambda t - \phi_B) \tag{3.20a}$$

$$h_2(t) = |A| \sin(t + \lambda t + \phi_A) + |B| \sin(t - \lambda t - \phi_B) \tag{3.20b}$$

where we denoted the complex numbers  $A$  and  $B$  in polar form  $C = |C| \exp(i\Phi_C)$ . Taking into account the initial conditions given in (3.13) we arrive to the four independent solutions

$$\mathbf{h}_1(t) = (\cos \lambda t \cos t, -\sin \lambda t \cos t, -\cos \lambda t \sin t, \sin \lambda t \sin t)^T \tag{2.21a}$$

$$\mathbf{h}_2(t) = (\sin \lambda t \cos t, \cos \lambda t \cos t, -\sin \lambda t \sin t, -\cos \lambda t \sin t)^T. \tag{3.21b}$$

Substituting these results in (3.12), we find that the matrix  $\Lambda$  can be written in blocks of  $2 \times 2$  matrices  $\lambda_k, 1 \leq k \leq 4$  such that

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \quad \lambda_k = \mu_k \mathbf{R} \tag{3.22a,b}$$

where  $\mu_1 = \mu_4 = \cos t$  and  $\mu_2 = -\mu_3 = \sin t$ .  $\mathbf{R}$  is a rotation matrix by an angle  $\theta = \lambda t$ , i.e.

$$\mathbf{R} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \tag{3.23}$$

Through these expressions it is possible to obtain immediately the constants of motion, which can be written down in the form of annihilation operators  $a_{k0} = (x_{k0} + ip_{k0})/\sqrt{2}$ :

$$a_{10}(t) = \exp(it)\{a_1 \cos(\lambda t) + a_2 \sin(\lambda t)\} \tag{3.24a}$$

$$a_{20}(t) = \exp(it)\{-a_1 \sin(\lambda t) + a_2 \cos(\lambda t)\} \tag{3.24b}$$

and their correspondent Hermitean conjugates. From these we built the annihilation operators (2.3) and found agreement with the results (2.13).

#### 4. Examples with time-dependent parameters

We study systems described by the Hamiltonian

$$H = \frac{1}{2} \sum_i \left( \frac{p_i^2}{m} + m\omega_0^2 x_i^2 \right) + \lambda(x_1 p_2 - x_2 p_1). \tag{4.1}$$

Through Noether's theorem we obtain linear time-dependent integrals of motion, and from these invariants, following [2-5], we can evaluate the evolution operator, the associated coherent states and the correlation matrices in  $pq$  and  $a^\dagger a$  spaces. In these calculations for quadratic Hamiltonians, the fundamental quantity is the symplectic matrix  $\Lambda$ , which relates linear time-dependent integrals of motion with the momentum and position operators. Next



we are going to consider the Hamiltonian (4.1) for several choices of the parameters  $m$  and  $\lambda$ , and determine the analytic expressions for the  $2 \times 2$  submatrices  $\lambda_k$  of  $\Lambda$ .

First, we consider an exponentially varying mass  $m = m_0 \exp\{\gamma t\}$ , and  $\lambda$  an arbitrary function of time. These parameters imply through (3.18) that  $\Omega^2 = \omega_0^2 - \gamma^2/4$ , so considering  $\gamma^2 < 4\omega_0^2$ , we solve equation (3.17), and using (3.19) together with the initial conditions (3.13), we determine that the matrix  $\Lambda$  of the system has the structure given in (3.22) but in this case the rotation angle is  $\theta = \int_0^t \lambda(\tau) d\tau$  and the  $\mu_k$  functions take the form

$$\mu_1 = \exp\{-\gamma t/2\}(\cos \Omega t + \frac{1}{2}\gamma(\sin \Omega t/\Omega)) \quad (4.2a)$$

$$\mu_2 = \exp\{\gamma t/2\}m_0\omega_0^2(\sin \Omega t/\Omega) \quad (4.2b)$$

$$\mu_3 = -\frac{1}{m_0} \exp\{-\gamma t/2\}(\sin \Omega t/\Omega) \quad (4.2c)$$

$$\mu_4 = \exp\{\gamma t/2\}(\cos \Omega t - \frac{1}{2}\gamma(\sin \Omega t/\Omega)). \quad (4.2d)$$

The case  $\gamma^2 > 4\omega_0^2$  is obtained from expressions (4.2) by making the replacement

$$\Omega \rightarrow i\Omega_1$$

with  $\Omega_1 = \sqrt{\gamma^2/4 - \omega_0^2}$ , and by means of the elementary relations  $\cos i\Omega_1 = \cosh \Omega_1$  and  $\sin i\Omega_1/i\Omega_1 = \sinh \Omega_1/\Omega_1$ . Finally, when  $\gamma = \pm 2\omega_0$ , the corresponding  $\lambda_k$  matrices are obtained from (4.2) by taking the limit when  $\Omega \rightarrow 0$ .

It is important to note that if we take  $\gamma = 0$ , the  $\mu_k$  functions take the form

$$\mu_1 = \cos \omega_0 t \quad (4.3a)$$

$$\mu_2 = m_0\omega_0 \sin \omega_0 t \quad (4.3b)$$

$$\mu_3 = -(1/m_0\omega_0) \sin \omega_0 t \quad (4.3c)$$

$$\mu_4 = \cos \omega_0 t \quad (4.3d)$$

from which, if we take  $m_0 = \omega_0 = 1$  and  $\lambda$  a constant, we recover the result given in the previous section.

Sometimes it is convenient to write the quantum invariants in the form of creation and annihilation operators, because the eigenfunctions of the integrals of motion  $\mathbf{A}(t) = (A_1(t), A_2(t))$  define solutions of the time-dependent Schrödinger equation or coherent-type states of the system:

$$\begin{pmatrix} \mathbf{A}(t) \\ \mathbf{A}^\dagger(t) \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix}. \quad (4.4)$$

The matrix  $\mathbf{M}$  is defined through the expression  $\mathbf{M} = g\Lambda g^{-1}$ , with the  $g$  matrix given by

$$g = \frac{1}{\sqrt{2}} \begin{pmatrix} i(l/\hbar)\mathbf{1}_2 & (1/l)\mathbf{1}_2 \\ -i(l/\hbar)\mathbf{1}_2 & (1/l)\mathbf{1}_2 \end{pmatrix} \quad (4.5)$$

and  $l = \sqrt{\hbar/m_0\omega_0}$  is the oscillator length. It is important to emphasize that  $\mathbf{M}$  is also a symplectic matrix. It is sometimes useful to write the matrix  $\mathbf{M}$  in terms of the  $2 \times 2$  matrices  $\mathbf{M}_k$ ,  $1 \leq k \leq 4$ :

$$\mathbf{M}_1 = \frac{1}{2} \left( \lambda_1 - im_0\omega_0\lambda_3 + \lambda_4 + \frac{i}{m_0\omega_0}\lambda_2 \right) \quad (4.6a)$$

$$\mathbf{M}_2 = \frac{1}{2} \left( -\lambda_1 + im_0\omega_0\lambda_3 + \lambda_4 + \frac{i}{m_0\omega_0}\lambda_2 \right) \quad (4.6b)$$

with  $M_3 = M_2^*$  and  $M_4 = M_1^*$ . If we consider  $m = 1$ , i.e.  $\gamma = 0$  and  $m_0 = 1$ , the operators (3.4) can be written

$$A_1(t) = \exp\{it\}\{a_1 \cos(\bar{\lambda}t) + a_2 \sin(\bar{\lambda}t)\} \tag{4.7a}$$

$$A_2(t) = \exp\{it\}\{-a_1 \sin(\bar{\lambda}t) + a_2 \cos(\bar{\lambda}t)\} \tag{4.7b}$$

where  $\bar{\lambda} = \int_0^t \lambda(\tau) d\tau/t$ , is the average of  $\lambda$  during the period of time  $t$ . Taking linear combinations as in (2.3) we arrive at the linear time-dependent invariants

$$\xi_+(t) = \exp[i(\omega_0 + \bar{\lambda})t]\xi_+ \tag{4.8a}$$

$$\xi_-(t) = \exp[i(\omega_0 - \bar{\lambda})t]\xi_- \tag{4.8b}$$

Proceeding as in (2.15) we obtain the Bragg-like conditions

$$k_1(\omega_0 + \bar{\lambda}) + k_2(\omega_0 - \bar{\lambda}) = 0 \quad \text{for } |\bar{\lambda}| > 1 \tag{4.9a}$$

$$k_1(\omega_0 + \bar{\lambda}) - k_2(\omega_0 - \bar{\lambda}) = 0 \quad \text{for } |\bar{\lambda}| < 1. \tag{4.9b}$$

Substituting (3.22b) into the expressions for the correlation matrices (A.6), given in the appendix and using the fact that  $\mathbf{R}$  is an orthogonal matrix, we have immediately that the dispersion matrices  $\sigma^2$  are independent on the parameter  $\lambda$ , and are diagonal:

$$\sigma_{pp}^2 = \frac{\hbar m_0 \omega_0}{2} \left\{ \frac{1}{(m_0 \omega_0)^2} \mu_2^2 + \mu_4^2 \right\} \mathbf{I}_2 \tag{4.10a}$$

$$\sigma_{pq}^2 = -\frac{\hbar}{2} \left\{ \frac{1}{m_0 \omega_0} \mu_2 \mu_1 + (m_0 \omega_0) \mu_4 \mu_3 \right\} \mathbf{I}_2 \tag{4.10b}$$

$$\sigma_{qq}^2 = \frac{\hbar}{2m_0 \omega_0} \{ \mu_1^2 + (m_0 \omega_0)^2 \mu_3^2 \} \mathbf{I}_2. \tag{4.10c}$$

Finally taking into account that for the considered system the  $\mu_k$  functions are given by (4.2) we obtain

$$\sigma_{pp}^2(t) = \frac{\hbar m_0 \omega_0}{2} \exp(\gamma t) \left[ 1 + \frac{\gamma^2}{2\Omega^2} \sin^2 \Omega t - \frac{\gamma}{2\Omega} \sin 2\Omega t \right] \mathbf{I}_2 \tag{4.11a}$$

$$\sigma_{qq}^2(t) = \frac{\hbar}{m_0 \omega_0} \exp(-\gamma t) \left[ 1 + \frac{\gamma^2}{2\Omega^2} \sin^2 \Omega t + \frac{\gamma}{2\Omega} \sin 2\Omega t \right] \mathbf{I}_2 \tag{4.11b}$$

$$\sigma_{pq}^2(t) = -\frac{\gamma \hbar \omega_0}{2\Omega^2} \sin^2 \Omega t \mathbf{I}_2. \tag{4.11c}$$

For the values of the parameters  $\gamma = 0.1$  and  $\omega_0$  equal to 1, 1/20 and 1/30, the behaviour of the dispersion matrices is illustrated in figures 1, 2 and 3, respectively. In these figures, we observe that there is squeezing for the coordinates and stretching for the momenta. It is also interesting to note that  $\sigma_{pq}$  is a negative function; because it is not identical to zero, there is correlation between the coordinates and the momenta. If we reverse the sign of  $\gamma$ , the dispersion for the coordinates and momenta are interchanged, and  $\sigma_{pq}$  becomes positive. In particular, for the case  $\gamma^2 < 4\omega_0^2$  shown in figure 1, the dispersion matrix  $\sigma_{pq}$  is a negative

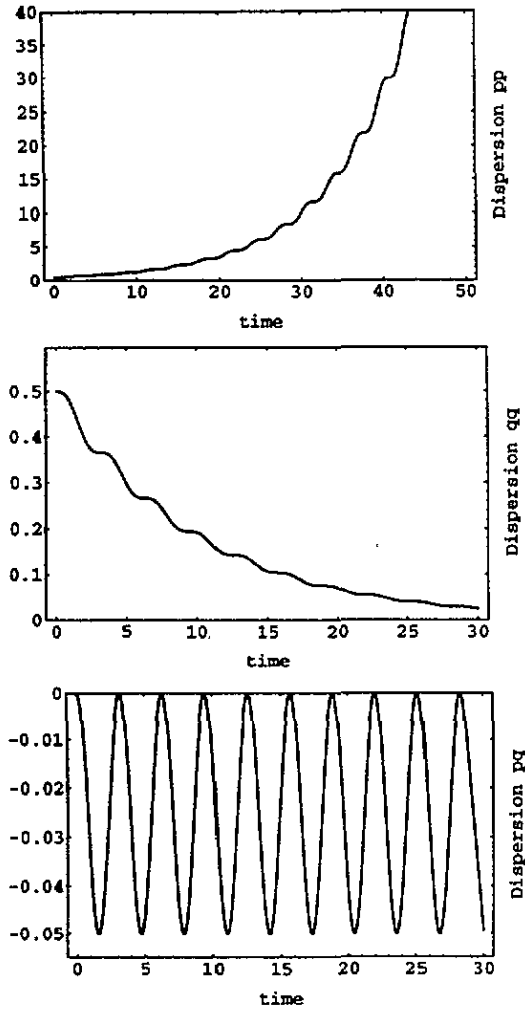


Figure 1. Dispersion matrix behaviour in coordinates and momenta space given by equation (4.14), for the case  $\gamma = 0.1$ , and  $m_0 = 1 = \omega_0$ .

oscillating function. Finally, the change in mass as a function of time, in arbitrary units, is displayed in figure 4.

Now we consider a varying mass of the form

$$m(t) = \begin{cases} m_0 & t \leq 0 \\ m_0 \cosh^2 \Omega_0 t & 0 \leq t \leq T \\ m_0 \{ \Omega_0 (t - T) \sinh \Omega_0 T + \cosh \Omega_0 T \}^2 & T \leq t \end{cases} \quad (4.12)$$

which is substituted into (3.18). We solve the ordinary differential equation (3.17) for the indicated time ranges, and obtain

$$w(t) = \begin{cases} A \exp(i\omega_0 t) + B \exp(-i\omega_0 t) & t \leq 0 \\ C \exp(i\tilde{\Omega} t) + D \exp(-i\tilde{\Omega} t) & 0 \leq t \leq T \\ F \exp(i\omega_0 t) + G \exp(-i\omega_0 t) & T \leq t \end{cases}$$

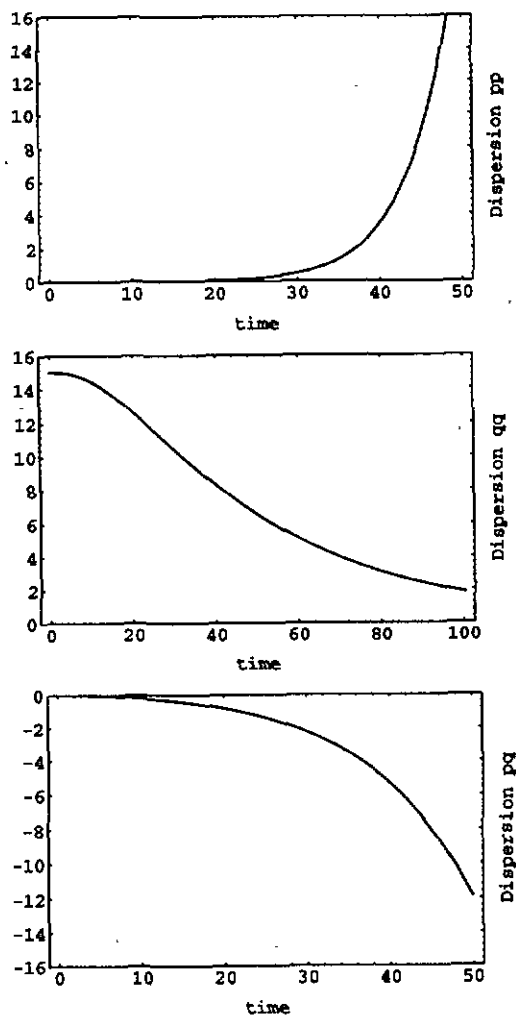


Figure 2. Dispersion matrices behaviour in coordinates and momenta space given by equation (4.14) in the limit when  $\gamma^2 \rightarrow 4\omega_0^2$ . The parameters used are  $\gamma = 0.1$  and  $m_0 = 1$ .

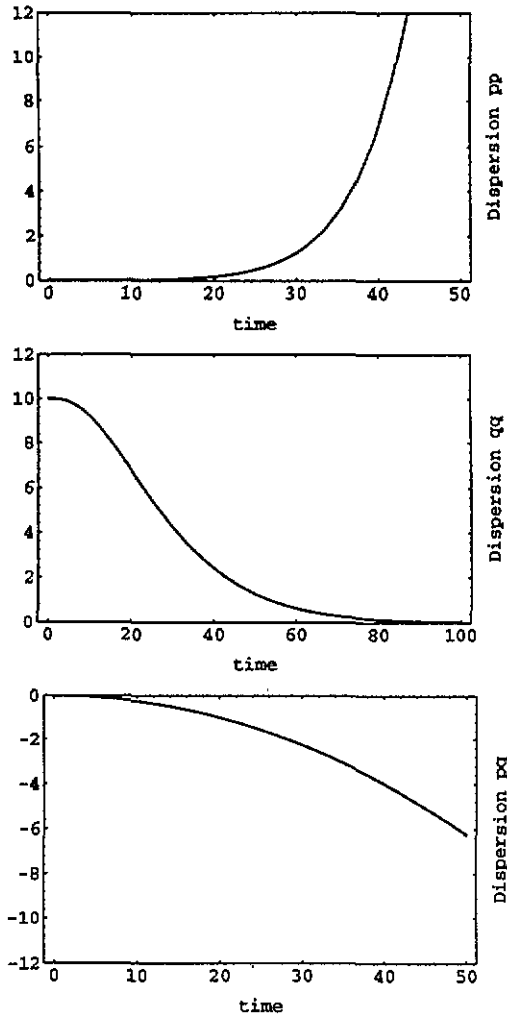
where  $\tilde{\Omega} \equiv \sqrt{\omega_0^2 - \Omega_0^2}$ . By asking for continuity conditions for the function  $w(t)$  and its derivative in  $t = 0$  and  $t = T$ , we find the following relations between the constants:

$$C = \frac{A+B}{2} + \frac{A-B}{2\tilde{\Omega}} \quad D = \frac{A+B}{2} - \frac{A-B}{2\tilde{\Omega}}$$

$$F = \exp(-i\omega_0 T) \left[ \frac{A+B}{2} \left( \cos \tilde{\Omega} T + i \frac{\tilde{\Omega}}{\omega_0} \sin \tilde{\Omega} T \right) + \frac{A-B}{2} \left( \cos \tilde{\Omega} T + i \frac{\tilde{\omega}_0}{\tilde{\Omega}} \sin \tilde{\Omega} T \right) \right]$$

$$G = \exp(i\omega_0 T) \left[ \frac{A+B}{2} \left( \cos \tilde{\Omega} T - i \frac{\tilde{\Omega}}{\omega_0} \sin \tilde{\Omega} T \right) - \frac{A-B}{2} \left( \cos \tilde{\Omega} T - i \frac{\tilde{\omega}_0}{\tilde{\Omega}} \sin \tilde{\Omega} T \right) \right].$$

Substituting the constants  $C, D, F$  and  $G$ , into the relation for  $w(t)$ , it is straightforward



**Figure 3.** Dispersion matrix behaviour in coordinates and momenta space given by the analytic continuation of equation (4.14) when  $\gamma^2 > 4\omega_0^2$ . The parameters are  $\gamma = 0.1, m_0 = 1,$  and  $\omega_0 = 1/30$ .

but lengthy to build the general solutions (3.19) for (3.9). The constants  $A$  and  $B$  are determined through the initial conditions (3.13).

This new system will have a  $\Lambda$  matrix, which has three different functional forms according to the time intervals indicated in equation (4.12); however, in all cases the  $\lambda_k$  matrices have the structure indicated in equation (3.22), with an  $\mathbf{R}$  matrix identical to the one of the previous example:

(i) The functions  $\mu_k$  for  $t \leq 0$  are obtained by taking  $\gamma = 0$  in expressions (4.3) and, similarly, the corresponding dispersion matrix is obtained through the equation (4.11).

(ii) For  $0 \leq t \leq T$ , we have to put the corresponding solutions (3.19) into the general relation (3.12). The resulting  $\mu_k$  functions are

$$\mu_1 = \cos \bar{\Omega}t / \cosh \Omega_0 t \tag{4.13a}$$

$$\mu_2 = m_0 \{ \bar{\Omega} \cosh \Omega_0 t \sin \bar{\Omega}t + \Omega_0 \cos \bar{\Omega}t \sinh \Omega_0 t \} \tag{4.13b}$$

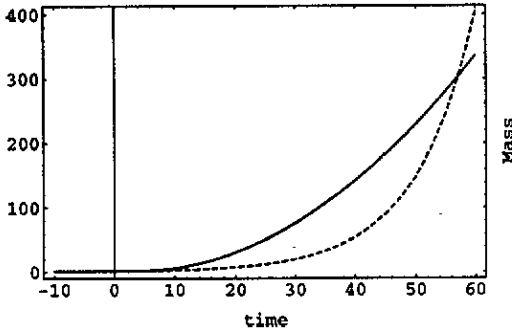


Figure 4. Time dependence of the mass: the broken curve corresponds to  $m(t) = m_0 \exp(\gamma t)$ , and the full curve to equation (4.15). The parameters used to display the plot are  $m_0 = 1$ ,  $\gamma = 0.1$ ,  $\Omega_0 = 0.15$ , and  $T = 10$ .

$$\mu_3 = -\sin \tilde{\Omega} t / m_0 \tilde{\Omega} \cosh \Omega_0 t \tag{4.13c}$$

$$\mu_4 = [\cosh \Omega_0 t \cos \tilde{\Omega} t - (\Omega_0 / \tilde{\Omega}) \sin \tilde{\Omega} t \sinh \Omega_0 t]. \tag{4.13d}$$

Substituting these into expressions (4.10) we arrive at the correlation matrices for  $0 \leq t \leq T$  associated with the system (4.1), with a mass parameter of the form (4.12):

$$\begin{aligned} \sigma_{pp}^2 = \frac{\hbar m_0 \omega_0}{4} \left\{ \frac{\tilde{\Omega}^2}{\omega_0^2} - \frac{\Omega_0^2}{\tilde{\Omega}^2} \sin^2(\tilde{\Omega} t) - \frac{\Omega_0^3}{\tilde{\Omega} \omega_0^2} \sinh(2\Omega_0 t) \sin(2\tilde{\Omega} t) \right. \\ \left. + \cosh(2\Omega_0 t) \left[ \sin^2(\tilde{\Omega} t) + \frac{\omega_0^2}{\tilde{\Omega}^2} \cos^2(\tilde{\Omega} t) \right] \right\} \mathbf{I}_2 \end{aligned} \tag{4.14a}$$

$$\sigma_{qq}^2 = \frac{\hbar}{2m_0 \omega_0} \operatorname{sech}^2(\Omega_0 t) \left[ \frac{\omega_0^2}{\tilde{\Omega}^2} - \frac{\Omega_0^2}{\tilde{\Omega}^2} \cos^2(\tilde{\Omega} t) \right] \mathbf{I}_2 \tag{4.14b}$$

$$\sigma_{pq}^2 = \frac{\hbar}{2} \left\{ \frac{\Omega_0^2}{2\omega_0 \tilde{\Omega}} \sin(2\tilde{\Omega} t) - \left[ \frac{\omega_0 \Omega_0}{\tilde{\Omega}^2} - \frac{\Omega_0^3}{\tilde{\Omega}^2 \omega_0} \cos^2(\tilde{\Omega} t) \right] \tanh(\Omega_0 t) \right\} \mathbf{I}_2. \tag{4.14c}$$

For  $t \geq T$  the  $\mu_k$  functions are given by the following cumbersome expressions:

$$\mu_1 = \frac{1}{\sqrt{m/m_0}} \left\{ \cos \tilde{\Omega} T \cos[\omega_0(t - T)] - \frac{\tilde{\Omega}}{\omega_0} \sin \tilde{\Omega} T \sin[\omega_0(t - T)] \right\} \tag{4.15a}$$

$$\begin{aligned} \mu_2 = \frac{\Omega_0 m_0}{2} \sinh(\Omega_0 T) \left\{ \left[ 1 - \frac{\tilde{\Omega}}{\omega_0} \right] \cos(-\omega_0(t - T) + \tilde{\Omega} T) \right. \\ \left. + \left[ 1 + \frac{\tilde{\Omega}}{\omega_0} \right] \cos(\omega_0(t - T) + \tilde{\Omega} T) \right\} \\ + \frac{m_0 \omega_0}{2} \{ \cosh(\sinh(\Omega_0 T)) \} \left\{ \left[ 1 + \frac{\tilde{\Omega}}{\omega_0} \right] \sin(\omega_0(t - T) + \tilde{\Omega} T) \right. \\ \left. - \left[ 1 - \frac{\tilde{\Omega}}{\omega_0} \right] \sin(-\omega_0(t - T) + \tilde{\Omega} T) \right\} \end{aligned} \tag{4.15b}$$

$$\mu_3 = -\frac{1}{m_0\omega_0} \frac{1}{\sqrt{m/m_0}} \left\{ \frac{\omega_0}{\tilde{\Omega}} \sin \tilde{\Omega}T \cos[\omega_0(t-T)] + \cos \tilde{\Omega}T \sin[\omega_0(t-T)] \right\} \quad (4.15c)$$

$$\begin{aligned} \mu_4 = \frac{\Omega_0}{2\tilde{\Omega}} \sinh(\Omega_0 T) & \left\{ - \left[ 1 + \frac{\tilde{\Omega}}{\omega_0} \right] \sin(\omega_0(t-T) + \tilde{\Omega}T) \right. \\ & + \left[ 1 - \frac{\tilde{\Omega}}{\omega_0} \right] \sin(\omega_0(t-T) - \tilde{\Omega}T) \left. \right\} \\ & + \frac{\omega_0}{2\tilde{\Omega}} \{ \cosh(\Omega_0 T) + \Omega_0(t-T) \sinh(\Omega_0 T) \} \left\{ \left[ 1 + \frac{\tilde{\Omega}}{\omega_0} \right] \cos(\omega_0(t-T) + \tilde{\Omega}T) \right. \\ & \left. - \left[ 1 - \frac{\tilde{\Omega}}{\omega_0} \right] \cos(\omega_0(t-T) - \tilde{\Omega}T) \right\}. \end{aligned} \quad (4.15d)$$

Substituting these results into equations (4.10) we arrive at the correlation matrices for  $t \geq T$ .

To illustrate this case we make a specific choice for the parameters of mass. We consider  $m_0 = \omega_0 = 1$ ,  $\Omega_0 = 0.15$  and  $T = 10$ , which is displayed in figure 4. The behaviour of the dispersion matrices for this selection of parameters are shown in figure 5. Although the change in mass with respect to time is different from that in the previous example (cf figure 4), the general trends for the correlation matrices are similar. For example,  $\sigma_{pp}$  is an increasing function of time starting from its minimum value at  $t \leq 0$ , and there is squeezing for the  $\sigma_{qq}$ . The main difference appears in the correlation  $\sigma_{pq}$ : in this case around the axis  $\sigma_{pq} = 0$  it is an oscillating function for  $t$  large enough, while in the previous cases it was negative or zero for any time.

### 5. Coherent and Fock States

Now we are going to build a general expression for the coherent-like states of the studied examples. This is carried out by solving the differential equations

$$A(t)\Phi_0(q, t) = 0 \quad (5.1)$$

where

$$A(t) = \lambda_p p + \lambda_q q. \quad (5.2a)$$

The  $\lambda_p$  and  $\lambda_q$  are given in terms of the  $\lambda_k$  matrices by

$$\lambda_p = \frac{1}{\sqrt{2\hbar m_0 \omega_0}} (i\lambda_1 + m_0 \omega_0 \lambda_3) \quad (5.2b)$$

$$\lambda_q = \sqrt{m_0 \omega_0 / 2\hbar} \left( -\frac{i}{m_0 \omega_0} \lambda_2 + \lambda_4 \right). \quad (5.2c)$$

The solution of (5.1) yields the vacuum state of the physical system, which takes the form

$$\Phi_0(q, t) = c(t) \exp \left\{ -\frac{i}{2\hbar} q \lambda_p^{-1} \lambda_q q \right\} \quad (5.3)$$

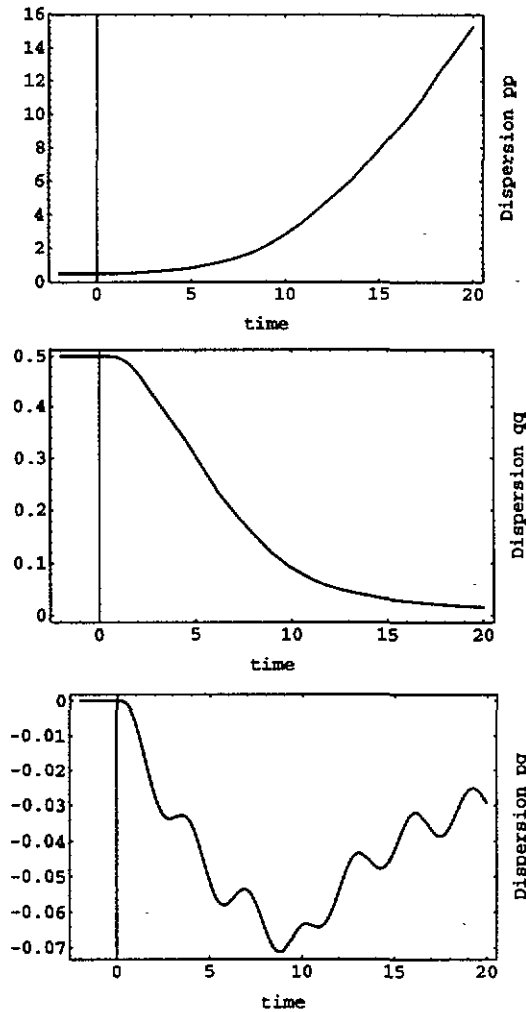


Figure 5. Dispersion matrix behaviour in coordinates and momenta space given in equation (3.20), for the case  $\Omega_0 = 0.15$ ,  $T = 10$  and  $m_0 = 1 = \omega_0$ .

where  $c(t)$  is the normalization constant. The phase of the wavefunction  $\Phi_0(q, t)$  is chosen to guarantee that it will be a solution of the time-dependent Schrödinger equation. Afterwards some calculations, the final expression for the ground-state wavefunction is

$$\Phi_0(q, t) = \frac{1}{\sqrt{2\pi\hbar^2\mu_p}} \exp\left\{-\frac{i}{2\hbar} \frac{\mu_q}{\mu_p} q \cdot q\right\}. \tag{5.4}$$

In all the studied examples the matrices  $\lambda_k$  are of the form (3.22), which was used in the previous expression, together with the definition of the functions  $\mu_p$  and  $\mu_q$  by the relations

$$\mu_p = \frac{1}{\sqrt{2\hbar m_0 \omega_0}} (i\mu_1 + m_0 \omega_0 \mu_3) \tag{5.5a}$$

$$\mu_q = \sqrt{m_0 \omega_0 / 2\hbar} \left(-\frac{i}{m_0 \omega_0} \mu_2 + \mu_4\right). \tag{5.5b}$$



Next, we build the unitary operator

$$\hat{D}(\alpha) = \exp\{\alpha \cdot A^\dagger - \alpha^* \cdot A\} \quad (5.6)$$

which is a power series expansion of integrals of motion and so it is also an invariant. Now we apply (5.6) to the vacuum wavefunction (5.4) and, after using a Baker–Campbell–Hausdorff formula [13], equation (5.1), and the action of the position and momentum operators, we obtain the wavefunction

$$\Phi_\alpha(q, t) = \exp\left\{-\frac{|\alpha|^2}{2} + \frac{1}{2} \frac{\mu_p^*}{\mu_p} \alpha \cdot \alpha + \frac{i}{\hbar \mu_p} \alpha \mathbf{R} q\right\} \Phi_0(q, t). \quad (5.7)$$

This wavefunction is the general expression for the coherent-like states in the coordinate representation of the discussed physical systems. By substituting the appropriate expressions for the functions  $\mu_p, \mu_q$  and  $\mathbf{R}$  into equation (5.7), we obtain the corresponding solutions for each case.

The wavefunction (5.7) can be expressed in terms of multi-dimensional Hermite polynomials [17] by making use of the relation

$$\exp\left(-\frac{1}{2} g_2 \alpha^* \cdot \alpha + g_3 \alpha^* \tilde{\mathbf{R}} \gamma\right) = \sum_{n_1, n_2=0}^{\infty} \frac{\alpha_1^{*n_1}}{n_1!} \frac{\alpha_2^{*n_2}}{n_2!} \mathbf{H}_{n_1, n_2}^{[g_2 \tilde{\mathbf{R}} \gamma]} \left(\frac{g_3}{g_2} \tilde{\mathbf{R}} \gamma\right). \quad (5.8)$$

Substituting this expression into (5.7) and using the form of the coherent-like states in the Fock-like representation, i.e.

$$\langle n_1, n_2, t | \alpha, t \rangle = \frac{\alpha_1^{n_1} \alpha_2^{n_2}}{\sqrt{n_1! n_2!}} \exp\left\{-\frac{|\alpha_1|^2}{2} - \frac{|\alpha_2|^2}{2}\right\} \quad (5.9)$$

we can obtain the Fock-like states in the coordinate representation:

$$\langle q | n_1, n_2 \rangle = \Phi_0(q, t) \mathbf{H}_{n_1, n_2}^{[-(\mu_p^*/\mu_p) \mathbf{l}_2]} \left(-\frac{i}{\hbar \mu_p^*} \mathbf{R} q\right). \quad (5.10)$$

The multi-dimensional Hermite polynomial can be rewritten as a product of two standard one-dimensional Hermite polynomials [17]:

$$\begin{aligned} \mathbf{H}_{n_1, n_2}^{[-(\mu_p^*/\mu_p) \mathbf{l}_2]} \left(-\frac{i}{\hbar \mu_p^*} \mathbf{R} q\right) &= \left(-\frac{\mu_p^*}{2\mu_p}\right)^{(n_1+n_2)/2} H_{n_1} \left(\frac{1}{\sqrt{2\hbar} |\mu_p|} [\cos \theta q_1 + \sin \theta q_2]\right) \\ &\times H_{n_2} \left(\frac{1}{\sqrt{2\hbar} |\mu_p|} [-\sin \theta q_1 + \cos \theta q_2]\right) \end{aligned} \quad (5.11)$$

where we have used the explicit expression of matrix  $\mathbf{R}$ . These Fock (5.10) and coherent (5.7)-like states are associated with the integrals of motion (5.2a) and, therefore, they represent squeezed and correlated states as shown in the previous section.

### 6. General quadratic case

In this section we show that the time-dependent invariants, linear in position and momentum, can be obtained through the Noether theorem procedure. Let us consider an arbitrary time-dependent multi-dimensional forced harmonic oscillator [5]

$$H = \frac{1}{2} Q_\alpha B_{\alpha\beta}(t) Q_\beta + C_\alpha Q_\alpha \tag{6.1}$$

where we have defined the vector

$$Q = \begin{pmatrix} p_1 \\ \vdots \\ p_n \\ q_1 \\ \vdots \\ q_n \end{pmatrix} \tag{6.2}$$

which denotes the  $n$  position and the  $n$  momentum operators and the matrices

$$B = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad C = \begin{pmatrix} F \\ G \end{pmatrix} \tag{6.3}$$

that characterize the quadratic form for the Hamiltonian. In equation (6.3),  $A, B, C, D$ , stand for  $n \times n$  matrices, and  $F, G$ , for  $n \times 1$  matrices. The Hermiticity of the Hamiltonian implies that the matrix  $B$  is symmetric, and this means the following symmetry conditions over the four constituents  $n \times n$  matrices:

$$A^t = A \quad B^t = C \quad D^t = D. \tag{6.4}$$

Expanding the Hamiltonian (6.1) we obtain

$$H = \frac{1}{2} (A_{\alpha\beta} p_\alpha p_\beta + 2B_{\alpha\beta} p_\alpha q_\beta + D_{\alpha\beta} q_\alpha q_\beta) + F_\alpha p_\alpha + G_\beta q_\beta \tag{6.5}$$

where the symmetry condition (6.4) was used.

Making a Legendre transformation and using the relation between velocities and momenta we obtain the Lagrangian of the system. Following the procedure indicated in section 3 to obtain the constants of motion for this system, let us propose an infinitesimal variation of coordinates given by

$$\delta q_\alpha = h_\alpha(t) \tag{6.6}$$

where  $h(t)$  is an arbitrary  $n$ -dimensional vector depending on time. The corresponding variation induced in the Lagrangian of the system can be rewritten as a total time derivative of a function  $\Omega$ , if the variation in the coordinates satisfies the differential equation

$$(\dot{h}_\alpha A_{\alpha\beta}^{-1})' + \dot{h}_\alpha (A^{-1} B)_{\alpha\beta} - (h_\alpha (C A^{-1})_{\alpha\beta})' - h_\alpha (C A^{-1} B - D)_{\alpha\beta} = 0 \tag{6.7}$$

which represents the homogeneous classical equation of motion for the coordinates of the system [18]. For this symmetry transformation the associated conserved quantities, according to Noether's theorem, are given by

$$J = \left( A_{\alpha\beta}^{-1} \dot{q}_\beta - (A^{-1} B)_{\alpha\beta} q_\beta - A_{\alpha\beta}^{-1} F_\beta \right) h_\alpha - A_{\alpha\beta}^{-1} \dot{h}_\alpha q_\beta + (C A^{-1})_{\alpha\beta} h_\alpha q_\beta + \int^t dt (A_{\alpha\beta}^{-1} \dot{h}_\alpha F_\beta + (C A^{-1})_{\alpha\beta} h_\alpha F_\beta + h_\alpha G_\alpha). \tag{6.8}$$

There are  $2n$  invariants because the system of equations (6.7) has  $2n$  independent solutions. These  $2n$  integrals of the motion can be rewritten in the following matrix form

$$\begin{pmatrix} \mathbf{p}_0(t) \\ \mathbf{q}_0(t) \end{pmatrix} = \Lambda(t) \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} + \Delta(t) \tag{6.9}$$

where  $\Lambda$  is a symplectic matrix in  $2n$  dimensions, which is given in terms of the solutions (6.7) and the matrices characterizing the physical system

$$\Lambda(t) = \begin{pmatrix} \mathbf{h}^{(1)} & (\mathbf{h}^{(1)}C - \dot{\mathbf{h}}^{(1)})A^{-1} \\ \mathbf{h}^{(2)} & (\mathbf{h}^{(2)}C - \dot{\mathbf{h}}^{(2)})A^{-1} \\ \vdots & \vdots \\ \mathbf{h}^{(2n)} & (\mathbf{h}^{(2n)}C - \dot{\mathbf{h}}^{(2n)})A^{-1} \end{pmatrix}. \tag{6.10}$$

The time-dependent column vector  $\Delta$  is given by

$$\Delta_k(t) = \int_0^t dt (A_{\alpha\beta}^{-1} \dot{h}_\alpha^{(k)} F_\beta + (CA^{-1})_{\alpha\beta} h_\alpha^{(k)} F_\beta + h_\alpha^{(k)} G_\alpha). \tag{6.11}$$

In expression (6.10), the superscript denotes the different solutions for system (6.7), and these vector solutions are written horizontally. The initial conditions for these solutions are

$$h_j^{(i)}(0) = \begin{cases} \delta_j^i & 1 \leq i, j \leq n \\ 0 & n + 1 \leq i, j \leq 2n \end{cases} \tag{6.12a}$$

and for their derivatives,

$$\dot{h}_j^{(i)}(0) = \begin{cases} C_{ij}(0) & 1 \leq i, j \leq n \\ -A_{ij}(0) & n + 1 \leq i, j \leq 2n \end{cases}. \tag{6.12b}$$

We have proved that linear time-dependent invariants for multi-dimensional oscillators can be obtained through Noether's theorem. This is achieved by considering a special variation which represents a translation along the classical trajectory of the quadratic system. To guarantee the existence of analytic solutions for the integrals of motion, it is only necessary to solve the classical equations of motion for this quadratic system. Because these invariants are linear in position and momentum its quantization is straightforward and then, by means of the theory of time-dependent invariants, the corresponding evolution operator and other relevant quantities can be determined.

### 7. Conclusions

In this work we have found the time-dependent integrals of the motion, that are linear in position and momentum, for the generalized two-dimensional harmonic oscillator using Noether's theorem. We consider, in general, varying mass and coupling strength in equation (2.1). The generators of the symmetry group, for an arbitrary time-dependent coupling strength, were obtained as a Bragg-like condition on the linear invariants. Using the model Hamiltonian (1.1) for specific choices of the time-dependent parameters we found that the four-dimensional symplectic matrix relating the invariants with the position and momentum operators is of the form indicated in equation (4.10), which implies that the correlation matrices are diagonal. The dispersion matrices obtained in the considered examples show squeezing and correlation. We have also constructed the corresponding solutions of the time-dependent Schrödinger equation. In particular, we wrote the coherent- and Fock-like states in the coordinate representation. Finally, we extended the procedure to find the time-dependent integrals of motion, that are linear in position and momentum, for the multi-dimensional quadratic Hamiltonian again using Noether's theorem. The time-dependent variations that give rise to these invariants satisfy the corresponding classical homogeneous equations of motion for the coordinates.

**Appendix. Correlation matrices**

We are going to show that, by means of the matrices  $\lambda_k$ , the  $2 \times 2$  correlation matrices for the position and momentum operators [5] can be immediately evaluated. Let us introduce the four-vector notation

$$Q = (p_1, p_2, x_1, x_2) \equiv (Q_\alpha) \quad 1 \leq \alpha \leq 4. \tag{A.1}$$

Then the dispersion matrix for the generalized coordinates  $Q_\alpha$  is

$$\sigma_{\alpha\beta}^2 = \frac{1}{2} \langle \{Q_\alpha, Q_\beta\} \rangle - \langle Q_\alpha \rangle \langle Q_\beta \rangle \tag{A.2}$$

where  $\{Q_\alpha, Q_\beta\}$  denotes the anticommutator of  $Q_\alpha$  and  $Q_\beta$ .

From the expression for the linear invariants in terms of the matrix  $\Lambda$ , it is straightforward to arrive at the following expression for the correlation matrices:

$$\sigma_{\alpha\beta}^2(t) = \Lambda_{\alpha\mu}^{-1}(t) \Lambda_{\beta\nu}^{-1}(t) \sigma_{\mu\nu}^2(0) \tag{A.3}$$

where  $\sigma^2(0)$  is the dispersion matrix for the initial conditions, i.e.

$$\sigma^2(0) = \frac{1}{2} \begin{pmatrix} \hbar m_0 \omega_0 \mathbf{I}_2 & 0 \\ 0 & \frac{\hbar}{m_0 \omega_0} \mathbf{I}_2 \end{pmatrix}. \tag{A.4}$$

Because  $\Lambda$  is a symplectic matrix, its inverse is given by the expression  $\Lambda^{-1} = -\Sigma \Lambda' \Sigma$  and substituting this result into the expression for  $\sigma_{\alpha\beta}^2(t)$ , we get

$$\sigma^2(t) = -\frac{1}{2} \Sigma \Lambda' \Sigma \sigma^2(0) \Sigma \Lambda \Sigma. \tag{A.5}$$

For the general quadratic case the matrix  $\Lambda$  has the form indicated in (3.22a) and, after substituting it into the last expression, we obtain the correlation matrices

$$\sigma_{pp}^2(t) = \frac{1}{2} \hbar m_0 \omega_0 \left( \frac{1}{(m_0 \omega_0)^2} \lambda_2^t \lambda_2 + \lambda_4^t \lambda_4 \right) \tag{A.6a}$$

$$\sigma_{pq}^2(t) = -\frac{\hbar}{2} \left( \frac{1}{m_0 \omega_0} \lambda_2^t \lambda_1 + m_0 \omega_0 \lambda_4^t \lambda_3 \right) \tag{A.6b}$$

$$\sigma_{qq}^2(t) = \frac{1}{2} \frac{\hbar}{m_0 \omega_0} (\lambda_1^t \lambda_1 + (m_0 \omega_0)^2 \lambda_3^t \lambda_3). \tag{A.6c}$$

For application to quantum optics, it is sometimes convenient to express the  $2 \times 2$  correlation matrices in terms of the creation and annihilation operators; that is,

$$\sigma_{aa}^2 = \frac{1}{2} \left( \frac{m_0 \omega_0}{\hbar} \sigma_{qq}^2 - \frac{1}{\hbar m_0 \omega_0} \sigma_{pp}^2 \right) + \frac{i}{\hbar} \sigma_{qp}^2 \tag{A.7a}$$

$$\sigma_{aa^\dagger}^2 = \frac{1}{2} \left( \frac{m_0 \omega_0}{\hbar} \sigma_{qq}^2 + \frac{1}{\hbar m_0 \omega_0} \sigma_{pp}^2 \right) \tag{A.7b}$$

$$\sigma_{a^\dagger a^\dagger}^2 = \frac{1}{2} \left( \frac{m_0 \omega_0}{\hbar} \sigma_{qq}^2 - \frac{1}{\hbar m_0 \omega_0} \sigma_{pp}^2 \right) + \frac{i}{\hbar} \sigma_{qp}^2. \tag{A.7c}$$

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